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# The Landau Problem on Compact Intervals and Optimal Numerical Differentiation

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## 1. SUMMARY

Let $m$ and $n$ be given integers, $0<m<n$. Let $f(x)$ be a real- or complex-valued function of a real variable $x$ on an interval $I$ such that $f^{(n-1)}(x)$ is absolutely continuous and $f^{(n)}(x)$ is bounded.

The Landau problem is estimating an intermediate derivative $f^{(m)}(x)$ when bounds for $f(x)$ and $f^{(n)}(x)$ are given. In this paper we present uniform bounds for $f^{(m)}(x)$ in terms of uniform bounds of $f(x)$ and $f^{(n)}(x)$. This improves earlier bounds given by H. Cartan by, roughly, a factor of $1/(e4^m)$.

Our method is based on the approximation of $f^{(m)}(x)$ by the $m$th derivative of a polynomial interpolating $f(x)$ at $n$ points in $I$. A technique to study the sign variations of the Peano kernel earlier used by us, Schönhage, and Schneider is developed further. We also use results by Gusev and by Rivlin.

Our method enables us to get estimates of the truncation error and of the magnification of errors in the values employed for $f(x)$ in such approximations.

## 2. INTRODUCTION

Let $m$ and $n$ be integers, $0<m<n$. Let $f^{(n-1)}(x)$ be absolutely continuous and $f^{(n)}(x)$ bounded on a compact interval $I$ of the real axis. We can, without loss of generality, specify $I$ to be $[0, 1]$ or $[-1, 1]$.

Let $I=[0, 1]$ and let $U=\{x_1, x_2, ..., x_n\}$ be a set of $n$ points in $I$ with $0=x_1<x_2<\cdots<x_n=1$. Let $\|f\|$ denote the essential supremum of $|f(x)|$ when $x$ belongs to $I$.



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Moreover let

$$\omega(x)=\omega(x, U)=\prod_{i=1}^{n}(x-x_i) \tag{2.1}$$

$$\phi_j(x)=\frac{\omega(x)}{(x-x_j)}, \qquad j=1, 2, ..., n \tag{2.2}$$

$$l_j(x)=\frac{\phi_j(x)}{\phi_j(x_j)}, \qquad j=1, 2, ..., n \tag{2.3}$$

$$L(x)=L_f(x, U)=\sum_{i=1}^{n} l_i(x) f(x_i). \tag{2.4}$$

Then $L(x)$ is the Lagrangian interpolation polynomial collocating with $f(x)$ at the set $U$.

It is well known that the difference between the derivatives of $f(x)$ and $L(x)$ may be written in the form

$$f^{(m)}(x)-L^{(m)}(x)=E_{n,m}(x), \tag{2.5}$$

where the remainder may be represented with a Peano kernel as

$$E_{n,m}(x)=\int_I f^{(n)}(t)\, K_x(t)\, dt. \tag{2.6}$$

The kernel can be written explicitly in the form

$$K_x(t)=K_x(t, U)=\frac{(x-t)_+^{n-1-m}}{(n-1-m)!}-\frac{1}{(n-1)!}\sum_{i=1}^{n} l_i^{(m)}(x)(x_i-t)_+^{n-1}. \tag{2.7}$$

Here $(x-t)_+=(x-t)$ when $x\geqslant t$ and is 0 elsewhere.

See for instance Powell [6], Kallioniemi [4], or Schönhage [12].

To emphasize a functions dependence of some of its variables we sometimes add or drop variables in our notations. It should be clear from the context what we mean.

Every choice of set $U$ will give us an upper bound of intermediate derivatives. By (2.5) and (2.6) we get

$$|f^{(m)}(x)|\leqslant M_0\sum_{i=1}^{n}|l_i^{(m)}(x, U)|+M_n\int_I |K_x(t, U)|\, dt, \tag{2.8}$$

where

$$M_0=M_{0U}=\operatorname{Max}|f(x_i)|, \qquad x_i\in U$$

and

$$M_n = M_{nI} = \|f^{(n)}\|.$$

The sum in the right-hand side of (2.8) gives an upper bound of the magnification of errors in the true values $f(x_i)$ used in Lagrangian numerical differentiation while the second term gives an upper bound of the truncation error. With this notation the Landau problem on finite intervals is closely connected with optimal choices of points in Lagrangian numerical differentiation.

Following Salzer [9] and Rivlin [7] we say that our formula (2.5) is optimally stable if the set $U$ is chosen such that it minimizes $\lambda_{m,n}(x, U)$ where

$$\lambda_{m,n}(x, U) = \sum_{i=1}^{n} |l_i^{(m)}(x, U)|. \tag{2.9}$$

Rivlin [7] proved the equivalence

$$\inf_{U} \lambda_{m,n}(x, U) = \max_{p \in P_{n-1}} |p^{(m)}(x)| = p_x^{(m)}(x), \tag{2.10}$$

where $P_{n-1}$ is the set of algebraic polynomials of degree $\leqslant n-1$ and with absolute value $\leqslant 1$ on $I$.

The extremal polynomials $p_x$ are known. See Gusev [3] and Rivlin [7]. The optimal set $U$ is for every $x$ in $I$ the set of $n$ points where the extremal polynomial $p_x$ attains the maximum of its modulus.

According to Gusev [3] there is a subset of $I$ of measure $m/(n-1)$ where $p_x$ equals the shifted Chebyshev polynomial $T_{n-1}^*(x) = \cos(n-1)\arccos(2x-1)$. Moreover

$$\|p_x^{(m)}\| \leqslant T_{n-1}^{*(m)}(1). \tag{2.11}$$

The set

$$C = \{x \mid |T_{n-1}^*(x)| = 1\} \tag{2.12}$$

is thus optimal in $I$ with respect to global stability. This set of points has also other advantages in numerical differentiation as pointed out by Salzer [10]. The computational effort needed to calculate the derivatives $L^{(m)}(x, U)$ can be facilitated when $U = C$.

When it comes to truncation error the set $C$ above is not optimal. The truncation error is given by our formulas (2.5) and (2.6). From (2.6) we infer

$$|E_{n,m}(x, U)| \leqslant M_n \int_I |K_x(t, U)|\, dt = M_n \mu_{n,m}(x, U). \tag{2.13}$$

For a given $x$ in $I$ the truncation error tends to zero when the points in $U$ tend to that $x$. Hence to be meaningful the truncation error has to be minimized either together with the magnifications of roundoff errors or in some global sense. In a paper by Schönhage [12] a dominant part of the truncation error is minimized with respect to some weighted $L_2$-norms in the case $m = 1$.

In this paper we say that the set $U$ is optimal with respect to truncation error if it minimizes the maximum of the truncation error in the regarded class of functions with bounded $n$th derivatives, i.e., if it minimizes $\|\mu_{n,m}(x, U)\|$.

If we let $f(x) = \omega(x)$ in the formulas (2.5) and (2.6) we immediately get

$$\mu_{n,m}(x, U) \geqslant \left| \frac{\omega^{(m)}(x, U)}{n!} \right|. \tag{2.14}$$

If the kernel $K_x(t)$ has constant sign when $t \in [0, 1]$ there will be equality in (2.14).

With ideas taken from Gusev [3] we will prove that there is equality in (2.14) when $x$ belongs to a subset of $I$ of measure $1 - m/(n-1)$, independently of the set $U$. If $x$ belongs to the remaining parts of $I$ the kernel $K_x(t)$ has one change of sign in [0, 1] and there cannot be equality in (2.14). Of course that must be the case when $\omega^{(m)}(x)$ equals or is very close to zero. The part of $I$ where (2.14) does not hold with equality consists of $n - m$ small intervals surrounding the zeros of $\omega^{(m)}(x)$. We can however give conditions on $U$ such that

$$\|\mu_{n,m}(x, U)\| = \left\| \frac{\omega^{(m)}(x, U)}{n!} \right\|. \tag{2.15}$$

The right-hand side of (2.15) is then not only "the dominant part" of the truncation error, it represents the upper bound of it. With this background we may limit our search to sets $U$ which minimize the right-hand side of (2.15) and satisfy (2.15). The problem of minimizing the truncation error is thus in some way analogue to the equivalence (2.10) for the roundoff error bounds.

The set $C$ which is optimal with respect to global stability is never optimal with respect to truncation error but it satisfies (2.15) and is a "good" choice. We will use that set in our inequality (2.8) to get uniform bounds of $f^{(m)}(x)$. These bounds will improve bounds given by Cartan [1].

In a paper by Pinkus [5] the Landau problem on finite intervals is solved in the sense that given bounds for $\|f\|$ and $\|f^{(n)}\|$ the function with largest $|f^{(m)}(x)|$ is described. This description is rather implicit and does not provide any general information about the size of the least upper bounds of the norms of intermediate derivatives.

## 3. Auxiliary Lemmas and Theorems

In our paper [4] we studied the sign variation of the functions $\phi_i^{(m)}(x)$ and of the kernel $K_x(t)$. In this section we go a bit further in such studies. We begin with some lemmas which can be found in [4].

Lemma 3.1. *Let $\alpha_1 = 0$ and let $\alpha_i$, $i = 2, ..., n-m$ be the successive zeros of $\phi_1^{(m)}(x)$. Let $\beta_{n-m} = 1$ and let $\beta_i$, $i = 1, 2, ..., n-m-1$ be the successive zeros of $\phi_n^{(m)}(x)$. Then we have*

$$\alpha_i < \beta_i < \alpha_{i+1} < \beta_{i+1}, \qquad i = 1, 2, ..., n-m-1$$

*and all the numbers $\phi_j^{(m)}(x)$, $j = 1, 2, ..., n$ have the same sign if and only if $x \in [\alpha_i, \beta_i]$ for some $i$, $1 \leqslant i \leqslant n-m$.*

Lemma 3.2. *The kernel $K_x(t)$ has constant sign in $[0, 1]$ when $x = 0$, when $x = 1$, and when $x \in [\beta_i, \alpha_{i+1}]$, $i = 1, 2, ..., n-m-1$. The kernel $K_x(t)$ changes sign at most once in $[0, 1]$ when $x \in ]\alpha_i, \beta_i[$, $i = 1, 2, ..., n-m$.*

Now let $A_U$ be the subset of $I$ where the numbers $\phi_j^{(m)}(x, U)$, $j = 1, 2, ..., n$, have the same sign and let $B_U$ be the subset of $I$ where the kernel $K_x(t)$ has constant sign in $[0, 1]$, i.e., let

$$A_U = \bigcup_{i=1}^{n-m} [\alpha_i, \beta_i] \tag{3.1}$$

$$B_U = \{0, 1\} \cup \bigcup_{i=1}^{n-m-1} [\beta_i, \alpha_{i+1}]. \tag{3.2}$$

Following ideas from Gusev [3] we get that the measures of $A_U$ and $B_U$ are independent of $U$. More precisely we have

Theorem 3.1. *The measures of $A_U$ and of $B_U$ are given by*

$$m(A_U) = \frac{m}{n-1} \tag{3.3}$$

$$m(B_U) = 1 - \frac{m}{n-1}. \tag{3.4}$$

*Proof.* We have

$$m(A_U)=\sum_{i=1}^{n-m}(\beta_i-\alpha_i)=\sum_{i=1}^{n-m}\beta_i-\sum_{i=1}^{n-m}\alpha_i \tag{3.5}$$

$$\phi_1(x)=\prod_{i=2}^{n}(x-x_i)=x^{n-1}-\left(\sum_{i=2}^{n}x_i\right)x^{n-2}+\cdots \tag{3.6}$$

$$\phi_n(x)=\prod_{i=1}^{n-1}(x-x_i)=x^{n-1}-\left(\sum_{i=1}^{n-1}x_i\right)x^{n-2}+\cdots. \tag{3.7}$$

From (3.6) and (3.7) we get

$$\phi_1^{(m)}(x)=\frac{(n-1)!}{(n-m-1)!}\left[x^{n-m-1}-\left(\frac{n-m-1}{n-1}\sum_{i=2}^{n}x_i\right)x^{n-m-2}+\cdots\right] \tag{3.8}$$

$$\phi_n^{(m)}(x)=\frac{(n-1)!}{(n-m-1)!}\left[x^{n-m-1}-\left(\frac{n-m-1}{n-1}\sum_{i=1}^{n-1}x_i\right)x^{n-m-2}+\cdots\right]. \tag{3.9}$$

By our definitions of $\alpha_i$ and $\beta_i$ we also have

$$\begin{aligned}\phi_1^{(m)}(x)&=\frac{(n-1)!}{(n-m-1)!}\prod_{i=2}^{n-m}(x-\alpha_i)\\&=\frac{(n-1)!}{(n-m-1)!}\left[x^{n-m-1}-\sum_{i=2}^{n-m}\alpha_i x^{n-m-2}+\cdots\right]\end{aligned} \tag{3.10}$$

and in the same way

$$\phi_n^{(m)}(x)=\frac{(n-1)!}{(n-m-1)!}\left[x^{n-m-1}-\sum_{i=1}^{n-m-1}\beta_i x^{n-m-2}+\cdots\right]. \tag{3.11}$$

By identifying coefficients of $x^{n-m-2}$ we get

$$\sum_{i=2}^{n-m}\alpha_i=\frac{n-m-1}{n-1}\sum_{i=2}^{n}x_i \tag{3.12}$$

and

$$\sum_{i=1}^{n-m-1}\beta_i=\frac{n-m-1}{n-1}\sum_{i=1}^{n-1}x_i. \tag{3.13}$$

From our formulas (3.5), (3.12), and (3.13) we then get

$$m(A_U)=\beta_{n-m}+\frac{n-m-1}{n-1}\left(\sum_{i=1}^{n-1}x_i-\sum_{i=2}^{n}x_i\right)-\alpha_1=\frac{m}{n-1} \tag{3.14}$$

and the theorem is proved.

*Remark.* When $x \in B_U$ the kernel $K_x(t)$ has constant sign in $[0, 1]$ and thus (2.14) holds with equality in the set $B_U$.

In the interval $[\beta_i, \alpha_{i+1}]$, $1 \leqslant i \leqslant n-m-1$, the function $|\omega^{(m)}(x)|$ has exactly one maximum, and the same is true for the function $\mu_{n,m}(x)$. Hence the set $B_U$ includes all the local extrema of $|\omega^{(m)}(x)|$. It is our guess that in the interval $[\alpha_i, \beta_i]$ the function $\mu_{n,m}(x)$ has exactly one minimum and could be majorized by a straight line between the values at the endpoints and thus giving (2.15). We have so far not been able to prove or disprove such a statement. We can however give conditions on the set $U$ such that in $[\alpha_i, \beta_i]$, $1 \leqslant i \leqslant n-m$, the function $\mu_{n,m}(x)$ is majorized by the maximum of $\mu_{n,m}(0)$ and $\mu_{n,m}(1)$. Then the relation (2.15) will follow.

By repeating the arguments used in [4, 10, 11] we can examine the number of sign variations in the differences between different kernels $K_x(t)$. We give without proof the following lemma.

LEMMA 3.3. *The functions $K_0(t) \pm K_x(t)$ and $K_1(t) \pm K_x(t)$ change sign at most once in* $[0, 1]$.

Our interest in studying only differences of the kind given in the lemma above is motivated by the two following lemmas.

LEMMA 3.4. *When $|l_1^{(m)}(x)| \leqslant |l_1^{(m)}(1)|$ we have*

$$\int_I |K_x(t)|\, dt \leqslant \int_I K_1(t)\, dt \tag{3.15}$$

*and when $|l_n^{(m)}(x)| \leqslant |l_n^{(m)}(0)|$ we have*

$$\int_I |K_x(t)|\, dt \leqslant \int_I |K_0(t)|\, dt. \tag{3.16}$$

*Proof.* Since the proofs of (3.15) and (3.16) are almost identical we limit ourselves to prove (3.15). Suppose that $t > x_{n-1}$ and that $0 < x < t < 1$. Then we get by (2.7) that

$$K_x(t) = -\frac{1}{(n-1)!}\, l_n^{(m)}(x)(1-t)^{n-1} \tag{3.17}$$

while

$$K_1(t) = \frac{1}{(n-m-1)!}(1-t)^{n-m-1} - \frac{1}{(n-1)!}\, l_n^{(m)}(1)(1-t)^{n-1}. \tag{3.18}$$

Hence it follows that when $t$ is close to 1 we have

$$K_1(t) > |K_x(t)|. \tag{3.19}$$

We may also represent the kernel $K_x(t)$ by

$$K_x(t) = \begin{cases} -\dfrac{1}{(n-1)!} \sum_{x_i > t} l_i^{(m)}(x)(x_i - t)^{n-1} & \text{when} \quad t \geqslant x \\ \dfrac{1}{(n-1)!} \sum_{x_i < t} l_i^{(m)}(x)(x_i - t)^{n-1}, & \text{when} \quad t < x. \end{cases} \tag{3.20}$$

See for instance Schönhage [12].

Suppose now that $t < x_2$ and that $0 < t < x \leqslant 1$. By (3.20) it then follows that

$$K_x(t) = \frac{1}{(n-1)!} l_1^{(m)}(x)(-t)^{n-1}. \tag{3.21}$$

Hence we get that if $|l_1^{(m)}(x)| < l_1^{(m)}(1)$ the inequality (3.19) will hold also when $t$ is close to 0. Since $K_1(t) > 0$ in $]0, 1[$ it then follows by Lemma 3.3 that $K_1(t) > |K_x(t)|$ in $]0, 1[$ which proves (3.15).

In our next lemma the conditions in Lemma 3.4 will be examined.

LEMMA 3.5. *Let*

$$q(x) = q(x, U) = \prod_{i=2}^{n-1} (x - x_i). \tag{3.22}$$

*Let the numbers $\alpha_i$ and $\beta_i$, $1 \leqslant i \leqslant n-m$, be as in Lemma 3.1 and let $\gamma_i$, $i = 2, 3, ..., n-m-1$, be the successive zeros of $q^{(m)}(x)$. We then have*

$$\alpha_i < \gamma_i < \beta_i, \qquad 2 \leqslant i \leqslant n-m-1 \tag{3.23}$$

$$\phi_1^{(m)}(\gamma_i) = \phi_n^{(m)}(\gamma_i), \qquad 2 \leqslant i \leqslant n-m-1 \tag{3.24}$$

$$|\phi_1^{(m)}(x)| \leqslant |\phi_1^{(m)}(\gamma_i)|, \qquad \textit{when} \quad x \in [\alpha_i, \gamma_i] \textit{ and } 2 \leqslant i \leqslant n-m-1 \tag{3.25}$$

$$|\phi_1^{(m)}(x)| \leqslant \phi_1^{(m)}(1), \qquad \textit{when} \quad x \in [\alpha_{n-m}, 1] \tag{3.26}$$

$$|\phi_n^{(m)}(x)| \leqslant |\phi_n^{(m)}(\gamma_i)|, \qquad \textit{when} \quad x \in [\gamma_i, \beta_i] \textit{ and } 2 \leqslant i \leqslant n-m-1 \tag{3.27}$$

$$|\phi_n^{(m)}(x)| \leqslant |\phi_n^{(m)}(0)|, \qquad \textit{when} \quad x \in [0, \beta_1]. \tag{3.28}$$

*Proof.* The function $|\phi_1^{(m)}(x)|$ is increasing in $[\alpha_{n-m}, 1]$ which gives (3.26). The function $|\phi_n^{(m)}(x)|$ is decreasing in $[0, \beta_1]$ giving (3.28). Suppose now that $x \in [\alpha_i, \beta_i]$ for some $i$, $2 \leqslant i \leqslant n-m-1$. Then $\phi_1^{(m)}(x)$ and $\phi_n^{(m)}(x)$

have the same sign. Since $\phi_1^{(m)}(\alpha_i)=\phi_n^{(m)}(\beta_i)=0$ there must be a point $\gamma$ in $[\alpha_i, \beta_i]$ where

$$\phi_1^{(m)}(\gamma)=\phi_n^{(m)}(\gamma). \tag{3.29}$$

By differentiating the relations

$$\phi_1(x)=(x-1)\,q(x)$$

and

$$\phi_n(x)=xq(x)$$

$m$ times we get

$$\phi_1^{(m)}(x)=(x-1)\,q^{(m)}(x)+mq^{(m-1)}(x) \tag{3.30}$$

and

$$\phi_n^{(m)}(x)=xq^{(m)}(x)+mq^{(m-1)}(x). \tag{3.31}$$

From Eqs. (3.29)–(3.31) it then follows that $q^{(m)}(\gamma)=0$, that is, $\gamma=\gamma_i$, proving (3.23) and (3.24).

We then get

$$\phi_1^{(m+1)}(\gamma)=(\gamma-1)\,q^{(m+1)}(\gamma) \tag{3.32}$$

and

$$\phi_n^{(m+1)}(\gamma)=\gamma q^{(m+1)}(\gamma). \tag{3.33}$$

Hence the numbers $\phi_1^{(m+1)}(\gamma)$ and $\phi_n^{(m+1)}(\gamma)$ are of opposite sign. Then $|\phi_1^{(m)}(x)|$ must be increasing in $[\alpha_i, \gamma_i]$ giving (3.25) while $|\phi_n^{(m)}(x)|$ must be decreasing in $[\gamma_i, \beta_i]$ giving (3.27), which concludes the proof of the lemma.

If the set $U$ is such that $|q^{(m-1)}(\gamma_i)|\leqslant q^{(m-1)}(1)$ we get by (3.30) that

$$|\phi_1^{(m)}(\gamma_i)|=m|q^{(m-1)}(\gamma_i)|\leqslant mq^{(m-1)}(1)=\phi_1^{(m)}(1). \tag{3.34}$$

Hence (3.15) follows by (3.25) for every $x$ in $[\alpha_i, \gamma_i]$, $2\leqslant i\leqslant n-m-1$, and by (3.26) for every $x$ in $[\alpha_{n-m}, 1]$.

If the set $U$ is such that $|q^{(m-1)}(\gamma_i)|\leqslant|q^{(m-1)}(0)|$ we get by (3.31) that

$$|\phi_n^{(m)}(\gamma_i)|=m|q^{(m-1)}(\gamma_i)|\leqslant m|q^{(m-1)}(0)|=|\phi_n^{(m)}(0)|. \tag{3.35}$$

Hence (3.16) follows by (3.27) for every $x$ in $[\gamma_i, \beta_i]$, $2\leqslant i\leqslant n-m-1$, and by (3.28) for every $x$ in $[0, \beta_1]$.

Hence we have proved the following theorem.

THEOREM 3.2. *Let* $K_x(t)$, $\omega(x)$, *and* $q(x)$ *be given by* (2.7), (2.1), *and* (3.22), *respectively. If the polynomial* $q(x)$ *satisfies*

$$|q^{(m-1)}(\gamma_i)| \leqslant \operatorname{Min}\{q^{(m-1)}(1), |q^{(m-1)}(0)|\}, \tag{3.36}$$

*where* $\gamma_i$, $2 \leqslant i \leqslant n-m-1$, *are the zeros of* $q^{(m)}(x)$, *we have*

$$\left\| \int_I |K_x(t)|\, dt \right\| = \left\| \frac{\omega^{(m)}}{n!} \right\|. \tag{3.37}$$

If we want to minimize the right-hand side of (3.37) the following lemma might be helpful.

LEMMA 3.6. *Let*

$$\Omega_n = \left\{ \omega(x) = \prod_{i=1}^{n} (x - x_i) \,\middle|\, 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots x_n \leqslant 1 \right\}. \tag{3.38}$$

*If there is a polynomial* $\omega^*_{n,m}$ *in* $\Omega_n$ *such that*

$$\|\omega^{*(m)}_{n,m}\| \leqslant \|\omega^{(m)}\| \tag{3.39}$$

*for every* $\omega \in \Omega_n$, *then* $\omega^*_{n,m}$ has to satisfy

$$\|\omega^{*(m)}_{n,m}\| = \omega^{*(m)}_{n,m}(1) = |\omega^{*(m)}_{n,m}(0)|. \tag{3.40}$$

*Proof.* Suppose on the contrary that $\omega^{*(m)}_{n,m}(1) < \|\omega^{*(m)}_{n,m}\|$. Then there must be an interval [0, a] with $a > 1$, such that

$$\max_{0 \leqslant x \leqslant 1} |\omega^{*(m)}_{n,m}(x)| = \max_{0 \leqslant x \leqslant a} |\omega^{*(m)}_{n,m}(x)|. \tag{3.41}$$

Let $\omega(x) = a^{-n}\omega^*_{n,m}(ax)$. Then $\omega \in \Omega_n$ and we get $\omega^{(m)}(x) = a^{m-n}\omega^{*(m)}_{n,m}(ax)$. Since $a > 1$ we get by (3.41) that $\|\omega^{(m)}\| < \|\omega^{*(m)}_{n,m}\|$ contradicting the definition of $\omega^*_{n,m}(x)$. The second equality in (3.40) is handled similarly.

Let $\Omega_n$ be defined by (3.38). According to compactness arguments there exists in $\Omega_n$ a polynomial $\omega^*_{n,m}(x)$, not necessarily unique, such that (3.39) holds. The extremal polynomial $\omega^*_{n,m}(x)$ may however have multiple zeros. In that case we have to allow some of the points of interpolation to coincide, which requires values of $f(x_i)$ together with some of its derivatives at such points. Hence in order to achieve optimality our concept of interpolation may have to be generalized.

The Chebyshev polynomials are extremal in the sense that with a given leading coefficient they have minimal norm. Since they have an enough number of zeros in the required range we get

$$\omega^*_{n,0}(x) = 2^{1-2n} T^*_n(x).$$

In a paper by Salzer [9] it is stated that "the dominant term in the remainder is minimal for arguments at the zeros of $r$th order integrals of Tschebyscheff polynomials specialized by addition of suitable $(r-1)$th degree polynomials chosen to produce real, distinct locations of points within or fairly close to the range of optimization." Salzer's variable "$r$" corresponds to our variable "$m$." Salzer's "dominant term" corresponds to the right-hand side of (2.14).

This statement by Salzer is, however, not true. We give here without proof

THEOREM 3.3. *Let $\omega^*_{n,m}(x)$ be extremal in the sense of* (3.39). *If $n$ is even we have*

$$\omega^*_{n,1}(x) = n2^{3-2n} \int_0^x T^*_{n-1}(t)\,dt.$$

*The interior zeros of $\omega^*_{n,1}(x)$ satisfy the condition* (3.36) *when $m=1$ and $n$ is even. If $n$ is odd there is no primitive of $cT^*_{n-1}(x)$ in the set $\Omega_n$.*

## 4. THE LANDAU PROBLEM ON BOUNDED INTERVALS

Let $C$ be defined by (2.12) and let in this section $\omega(x)=\omega(x, C)$. This set is optimal in the sense that it minimizes the sum in the right-hand side of (2.8). Thus it remains to estimate the integral in that inequality. To that end we have to see whether the conditions in the preceding section are satisfied by the set $C$.

We are going to use some well-known properties of the Chebyshev polynomials. For a reference see Rivlin [8]. Let now

$$c_n = (n-1)2^{2n-3}. \tag{4.1}$$

We then have

$$c_n\omega(x) = x(x-1)(T^*_{n-1})'(x). \tag{4.2}$$

The polynomial $T^*_{n-1}(x)$ satisfies the differential equation

$$2x(1-x)(T^*_{n-1})''(x) - (2x-1)(T^*_{n-1})'(x) + 2(n-1)^2\, T^*_{n-1}(x) = 0. \tag{4.3}$$

Moreover we have

$$\|(T^*_{n-1})^{(m)}\| \leqslant (T^*_{n-1})^{(m)}(1) = |(T^*_{n-1})^{(m)}(0)|. \tag{4.4}$$

Let

$$q(x, C) = c_n(T^*_{n-1})'(x). \tag{4.5}$$

By (4.5) and (4.4) we see that the condition (3.36) is satisfied by $q(x, C)$. We have also

LEMMA 4.1.

$$\|\omega^{(m)}(x, C)\| = \omega^{(m)}(1, C) = |\omega^{(m)}(0, C)|. \tag{4.6}$$

*Proof.* By differentiating Eq. (4.2) $m$ times we obtain

$$\begin{aligned} c_n\omega^{(m)}(x) = x(x-1)(T^*_{n-1})^{(m+1)}(x) + m(2x-1)(T^*_{n-1})^{(m)}(x) \\ + m(m-1)(T^*_{n-1})^{(m-1)}(x) \end{aligned} \tag{4.7}$$

and by differentiating (4.3) $m-1$ times we obtain

$$\begin{aligned} 2x(1-x)(T^*_{n-1})^{(m+1)}(x) - (2m-1)(2x-1)(T^*_{n-1})^{(m)}(x) \\ + 2[(n-1)^2 - (m-1)^2](T^*_{n-1})^{(m-1)}(x) = 0. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8) it then follows

$$2c_n\omega^{(m)}(x) = (2x-1)(T^*_{n-1})^{(m)}(x) + 2[(n-1)^2 + (m-1)](T^*_{n-1})^{(m-1)}(x). \tag{4.9}$$

Using (4.4) we infer that all the terms in the right-hand side of (4.9) have their greatest modulus when $x=1$ and when $x=0$, which proves the lemma.

From Theorem 3.2 and Lemma 4.1 now follows

THEOREM 4.1. *Let the set $C$ be defined by* (2.12). *Then we have*

$$\left\| \int_I |K_x(t, C)| \, dt \right\| \leq \frac{\omega^{(m)}(1, C)}{n!}. \tag{4.10}$$

The sum in (2.8) is now easy to estimate. We have

THEOREM 4.2. *Let $l_i(x, C)$ be defined by* (2.1)–(2.3). *Then*

$$\sum_{i=1}^{n} |l_i^{(m)}(x, C)| \leq (T^*_{n-1})^{(m)}(1). \tag{4.11}$$

This theorem can also be found in Rivlin [7]. We give however a proof, which is based upon a lemma due to Duffin and Schaeffer [2]. We formulate it for the interval $I = [0, 1]$.

LEMMA 4.2 (Duffin and Schaeffer). *If $f(z)$ is a polynomial of degree $n-1$ with real coefficients and satisfying $|f(x_i)| \leqslant 1$, $1 \leqslant i \leqslant n$, $x_i \in C$ then for every $m \geqslant 1$ and every $x$, $0 \leqslant x \leqslant 1$,*

$$|f^{(m)}(x+iy)| \leqslant |(T^*_{n-1})^{(m)}(1+iy)|. \tag{4.12}$$

*The equality occurs only if $f(z) = \pm T^*_{n-1}(z)$.*

*Proof of Theorem* 4.2. Let

$$f(x) = \sum_{i=1}^{n} \varepsilon_i l_i(x, C), \qquad \text{where} \quad |\varepsilon_i| = 1, \ 1 \leqslant i \leqslant n. \tag{4.13}$$

Then we have $f(x_i) = \varepsilon_i$, $1 \leqslant i \leqslant n$, and we get by Lemma 4.2 that for every $x$ in $I$

$$|f^{(m)}(x)| \leqslant (T^*_{n-1})^{(m)}(1). \tag{4.14}$$

Let now $x$ be a fixed point in $I$ and let

$$\varepsilon_i = \operatorname{sgn} l_i^{(m)}(x, C), \qquad 1 \leqslant i \leqslant n. \tag{4.15}$$

Then the left-hand side of (4.11) equals $f^{(m)}(x)$ and the theorem follows by (4.14).

*Remark.* If the values $l_i^{(m)}(x, C)$ have alternating signs, that is, if $x \in A_C$, then inequality (4.11) can be sharpened. If we in our formulas (2.5) and (2.6) let $f(x) = T^*_{n-1}(x)$ we get the remainder $E_{n,m}(x) = 0$ by which follows

$$\sum_{i=1}^{n} |l_i^{(m)}(x, C)| = |(T^*_{n-1})^{(m)}(x)|, \qquad x \in A_C. \tag{4.16}$$

By our Theorems 4.1 and 4.2 and formula (2.8) we will now get uniform bounds of intermediate derivatives.

We establish them in a general form.

THEOREM 4.3. *Let $f(x)$ be such that $f^{(n-1)}(x)$ is absolutely continuous and $f^{(n)}(x)$ exists almost everywhere and is bounded in $I = [0, a]$. Let $x_i \in C$ where $C$ is defined by* (2.12). *Moreover let*

$$M_{0C} = \max_{1 \leqslant i \leqslant n} |f(ax_i)| \tag{4.17}$$

*and*

$$M_{nI} = \operatorname*{ess\,sup}_{x \in I} |f^{(n)}(x)|. \tag{4.18}$$

*Then for every $x$ in $I$ and for every integer $m$, $1 \leqslant m \leqslant n$, we have*

$$|f^{(m)}(x)| \leqslant (T_{n-1}^{*})^{(m)}(1)(a^{-m}M_{0C} + b_{n,m}a^{n-m}M_{nI}), \tag{4.19}$$

*where*

$$b_{n,m} = \frac{m(2(n-1)^2 + m - 1)}{2^{2n-2}(n-1)(n-m)(n+m-2)n!}. \tag{4.20}$$

*Proof.* With a change of scale it is sufficient to prove the theorem when $a = 1$. Using the relation (see [8])

$$(T_{n-1}^{*})^{(k)}(1) = 2^{2k}\frac{k!}{(2k)!}(n-1)\frac{(n+k-2)!}{(n-k-1)!} \tag{4.21}$$

with $k = m$ and $k = m-1$ we get by (4.9) that

$$c_n\omega^{(m)}(1) = \frac{m(2(n-1)^2 + m - 1)}{2(n-m)(n+m-2)}(T_{n-1}^{*})^{(m)}(1). \tag{4.22}$$

Our formulas (2.8), (4.1), (4.10), (4.11), and (4.22) will then give (4.19) and conclude the proof.

If the interval $[0, a]$ is long enough we can replace the right-hand side of (4.19) by its minimum value with respect to $a$. To that end let

$$a' = \left(\frac{M_{0C}}{M_{nI}}\frac{m}{(n-m)b_{n,m}}\right)^{1/n}. \tag{4.23}$$

If we let $a = a'$ in (4.19) we get

$$|f^{(m)}(x)| \leqslant C_{n,m}(M_{0C})^{1-m/n}(M_{nI})^{m/n}, \tag{4.24}$$

where

$$C_{n,m} = (T_{n-1}^{*})^{(m)}(1)\frac{n}{n-m}\left[\left(\frac{n-m}{n}\right)b_{n,m}\right]^{m/n}. \tag{4.25}$$

With the aid of Stirlings formula

$$n! = a_n\sqrt{n}\left(\frac{n}{e}\right)^n \tag{4.26}$$

we can write

$$C_{n,m} = A_{n,m}B_{n,m}, \tag{4.27}$$

where

$$A_{n,m}=\frac{a_m a_{n+m}}{a_{n-m}a_{2m}}\sqrt{\frac{n+m}{2(n-m)}}\left(\frac{8}{a_n\sqrt{n}}\right)^{m/n}$$
$$\times\frac{n(n-1)}{(n+m)(n+m-1)}\left(\frac{2(n-1)^2+m-1}{2(n-1)(n+m-2)}\right)^{m/n} \tag{4.28}$$

and

$$B_{n,m}=\frac{(n+m)^{n+m}}{(4nm)^m(n-m)^{n-m}}. \tag{4.29}$$

The values $A_{n,m}$ are bounded. More precisely we have

$$A_{n,m}<\frac{2}{e}, \qquad 1\leqslant m\leqslant n, \tag{4.30}$$

while

$$\lim_{n\to\infty} A_{n,1}=\frac{2}{e}. \tag{4.31}$$

The proofs of these last two statements are placed at the end of this paper.

Estimates of the values $B_{n,m}$ are found in the literature. See, for instance, Stechkin [13]. We have

$$B_{n,m}<\left(\frac{e^2n}{4m}\right)^m, \qquad 1\leqslant m<n, \tag{4.32}$$

and

$$B_{n,m}<\left(\frac{2n}{n-m}\right)^{n-m}, \qquad 1\leqslant m<n, \tag{4.33}$$

whichever is preferable.

We will now present our estimates in a simplified form. To avoid the dependence of the interval length and of the numbers $x_i$, in the value $M_{0C}$ we now let

$$M_0=\max_{0\leqslant x\leqslant a}|f(x)| \tag{4.34}$$

and

$$M_n=\operatorname*{ess\,sup}_{0\leqslant x\leqslant a}|f^{(n)}(x)|. \tag{4.35}$$

THEOREM 4.4. *Let $M_0$ and $M_n$ be given by* (4.34)–(4.35). *Let $f(x)$, $a$, and the numbers $b_{n,m}$ be as in Theorem* 4.3. *Then for every $x$ in $[0, a]$ and every integer $m$, $1 \leqslant m \leqslant n$, we have*

$$|f^{(m)}(x)| \leqslant \frac{2}{e} B_{n,m}(M_0)^{1-m/n}(M'_n)^{m/n}, \tag{4.36}$$

*where*

$$M'_n = \max\left(M_n, \frac{mM_0}{a^n(n-m)b_{n,m}}\right) \tag{4.37}$$

*and the values $B_{n,m}$ are given by* (4.29)

*Remark.* The uniform bounds given by Cartan [1] are similar to (4.36) but are roughly $e4^m$ times greater than ours.

## 5. PROOFS OF SOME ESTIMATES IN SECTION 4

In this section we give detailed proofs of our formulas (4.30) and (4.31). In the proofs only elementary calculus is needed. We break down the proofs by some lemmas.

LEMMA 5.1. *Let the numbers $a_n$, $n \geqslant 1$, be defined by*

$$n! = a_n \sqrt{n}\left(\frac{n}{e}\right)^n. \tag{5.1}$$

*Then we have*

$$e = a_1 > a_2 > \cdots > a_n > a_{n+1} > \cdots > \sqrt{(2\pi)}. \tag{5.2}$$

*Proof.* Let ln denote the natural logarithm and let

$$g(n) = \ln \frac{a_{n+1}}{a_n} = 1 + \left(n + \frac{1}{2}\right) \ln \frac{n}{n+1}. \tag{5.3}$$

It then easily follows that $g''(n) < 0$ and that $g(n)$ tends to 0 when $n$ tends to infinity. Hence $g(n) < 0$, which proves that $a_n > a_{n+1}$. The last inequality in (5.2) follows from the well known fact that $a_n$ tends to $\sqrt{(2\pi)}$ when $n$ tends to infinity.

We now return to our formula (4.28). Let

$$D_{n,m} = \sqrt{\frac{n+m}{n-m}}\left(\frac{8}{a_n\sqrt{n}}\right)^{m/n} \frac{n(n-1)}{(n+m)(n+m-1)} \tag{5.4}$$

and

$$E_{n,m}=\left(\frac{2(n-1)^2+m-1}{2(n-1)(n+m-2)}\right)^{m/n}. \tag{5.5}$$

LEMMA 5.2. *Let the numbers* $D_{n,m}$ *be defined by* (5.4). *We then have that*

$$D_{n,m}\leqslant D_{n,n-1}<1.163 \qquad \textit{for every } m,\ 1\leqslant m\leqslant n-1, \tag{5.6}$$

*while*

$$D_{n,m}<1, \qquad \textit{when} \quad n\geqslant 2m\geqslant 2. \tag{5.7}$$

*Proof.* Let

$$h(n,m)=\ln D_{n,m}=\frac{m}{n}\ln\frac{8}{a_n\sqrt{n}}-\frac{1}{2}\ln(n^2-m^2)+\ln\frac{n(n-1)}{n+m-1}. \tag{5.8}$$

The second derivative of $h(n,m)$, with respect to $m$, is $>0$. Hence $h(n,m)$ attains its maximum values with respect to $m$ on the boundary. We have

$$h(n,1)<\frac{1}{n}\ln\frac{8}{\sqrt{2\pi n}}+\frac{1}{2}\ln\frac{n-1}{n+1}=h_1(n). \tag{5.9}$$

We get $h_1'(n)>0$ and hence $h_1(n)$ is increasing. Since $h_1(n)$ tends to 0 when $n$ tends to infinity we infer

$$h(n,1)<0. \tag{5.10}$$

When $m=n-1$ we get by (5.8) and (5.2) that

$$h(n,n-1)<\frac{n-1}{n}\ln\frac{8}{\sqrt{2\pi n}}-\frac{1}{2}\ln(2n-1)+\ln\frac{n}{2}=h_2(n). \tag{5.11}$$

The derivative of $h_2(n)$ is

$$\begin{aligned} h_2'(n)&=\frac{1}{n^2}\left(\ln\frac{8}{\sqrt{2\pi n}}-(n-1)\frac{1}{2}\right)-\frac{1}{2n-1}+\frac{1}{n}\\ &=\frac{1}{n^2}\left(\ln\frac{8}{\sqrt{2\pi n}}+\frac{n-1}{2(2n-1)}\right)=\frac{1}{n^2}h_3(n). \end{aligned} \tag{5.12}$$

The derivative of $h_3(n)$ is $<0$ when $n\geqslant 3$. Hence $h_2'(n)$ has at most one zero. We have

$$h_3(16)>0.016 \qquad \text{and} \qquad h_3(17)<-0.013$$

giving

$$h(n, n-1) < \max\{h_2(16), h_2(17)\} < 0.151. \tag{5.13}$$

Hence by (5.10) and (5.13) it follows that

$$h(n, m) \leqslant \max\{h(n, 1), h(n, n-1)\} < 0.151 \tag{5.14}$$

giving

$$D_{n,m} < \exp 0.151 < 1.163 \tag{5.15}$$

and thus proving (5.6).

In the same way (5.7) will follow if

$$h(2m, m) < 0. \tag{5.16}$$

By (5.8) and (5.2) we get

$$h(2m, m) < \frac{1}{2} \ln \frac{4}{\sqrt{m\pi}} - \frac{1}{2} \ln(3m^2) + \ln \frac{2m(2m-1)}{3m-1} = h_4(m). \tag{5.17}$$

The derivative of $h_4(m)$ is $<0$ and hence $h_4(m)$ is decreasing. We have then

$$h(2m, m) < h_4(1) < -0.14 \tag{5.18}$$

which concludes the proof of the lemma.

LEMMA 5.3. *Let $E_{n,m}$ be defined by* (5.5). *Then $E_{n,m}$ increases as a function of n and decreases as a function of m and satisfies*

$$E_{n,m} \leqslant E_{n,1} = 1, \qquad \textit{when} \quad 1 \leqslant m \leqslant n-1 \tag{5.19}$$

*while*

$$E_{n,m} \leqslant E_{2m,m} \leqslant E_{4,2} < 0.89, \qquad \textit{when} \quad 4 \leqslant n \leqslant 2m. \tag{5.20}$$

*Proof.* Let

$$f(n, m) = \ln E_{n,m} = \frac{m}{n} \ln \left( \frac{2(n-1)^2 + m - 1}{2(n-1)(n+m-2)} \right). \tag{5.21}$$

The derivative of $f(n, m)$ with respect to $m$ is $<0$ which gives (5.19). The derivative of $f(n, m)$ with respect to $n$ is $>0$ which gives the first part of (5.20).

On the other hand $E_{2m,m}$ is a decreasing function of $m$ which secures the remaining part of (5.20) and thus concludes the proof.

By (4.28), (5.4), and (5.5) we get

$$A_{n,m} = \frac{a_m a_{n+m}}{a_{n-m} a_{2m}} \frac{1}{\sqrt{2}} D_{n,m} E_{n,m}. \tag{5.22}$$

From (5.2) it follows

$$\frac{a_m a_{n+m}}{a_{n-m} a_{2m}} < 1, \qquad \text{when} \quad m < n < 2m. \tag{5.23}$$

By (5.6), (5.20), and (5.22) we then get

$$A_{n,m} < \frac{1}{\sqrt{2}} (1.163)(0.89) < 0.732 < \frac{2}{e}, \qquad \text{when} \quad m < n < 2m. \tag{5.24}$$

On the other hand we get from (5.19), (5.7), and (5.2) that

$$A_{n,m} < \frac{a_m a_{n+m}}{a_{n-m} a_{2m}} \frac{1}{\sqrt{2}} < \frac{a_m}{a_{2m}} \frac{1}{\sqrt{2}}, \qquad \text{when} \quad n \geqslant 2m. \tag{5.25}$$

We then get from (5.25) and (5.2) that

$$A_{n,m} < \frac{1}{\sqrt{2}} \frac{a_3}{\sqrt{2\pi}} = \frac{e^3}{9\sqrt{3\pi}} < 0.727 < \frac{2}{e}, \qquad \text{when} \quad n \geqslant 2m \geqslant 6. \tag{5.26}$$

We also have by (5.25) and (5.2) that

$$A_{n,2} < \frac{1}{\sqrt{2}} \frac{a_2}{a_4} = \frac{16}{3e^2} < \frac{2}{e}, \qquad \text{when} \quad n \geqslant 4 \tag{5.27}$$

and that

$$A_{n,1} < \frac{1}{\sqrt{2}} \frac{a_1}{a_2} = \frac{2}{e}, \qquad \text{when} \quad n \geqslant 2. \tag{5.28}$$

From our formulas (5.24)–(5.28) then (4.30) will follow.

We have $E_{n,1} = 1$ and we get from (5.4) that $D_{n,1}$ tends to 1 when $n$ tends to infinity. Since $a_{n+1}$ and $a_{n-1}$ have the same limit we then get by (5.22) that

$$\lim_{n \to \infty} A_{n,1} = \frac{a_1}{a_2} \frac{1}{\sqrt{2}} = \frac{2}{e} \tag{5.29}$$

which proves (4.31) and ensures that (4.30) cannot be refined.

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